SERIE 1 (Correction)

Exercice 1.

1.

$$\begin{aligned} z_1 &= \frac{3+6i}{3-4i} = \frac{(3+6i)(3+4i)}{(3-4i)(3+4i)} = \frac{9+12i+18i-24}{3^2-(4i)^2} \\ &= \frac{-15+30i}{9+16} = \frac{-15+30i}{25} \\ &= -\frac{3}{5} + \frac{6}{5}i \end{aligned}$$

2.

$$\begin{aligned} z_2 &= (\frac{1+i}{2-i})^2 = \frac{(1+i)^2}{(2-i)^2} = \frac{1+2i-1}{4-4i-1} \\ &= \frac{2i}{3-4i} = \frac{2i(3+4i)}{3^2-(4i)^2} = \frac{6i-8}{25} \\ &= -\frac{8}{25} + \frac{6}{25}i \end{aligned}$$

3.

$$z_3 = (-\frac{1}{2} + i\frac{\sqrt{3}}{2})^3 = (e^{\frac{2i\pi}{3}})^3 = e^{2i\pi} = 1.$$

4.

$$\begin{aligned} z_4 &= \frac{(1+i)^9}{(1-i)^7} = \frac{(\sqrt{2}(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}))^9}{(\sqrt{2}(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}))^7} = \frac{(\sqrt{2})^9(e^{\frac{i\pi}{4}})^9}{(\sqrt{2})^7(e^{\frac{-i\pi}{4}})^7} \\ &= (\sqrt{2})^2\frac{e^{\frac{i9\pi}{4}}}{e^{\frac{-i7\pi}{4}}} = 2e^{i(\frac{9\pi}{4}+\frac{7\pi}{4})} = 2e^{\frac{i16\pi}{4}} \\ &= 2e^{i4\pi} \end{aligned}$$

5.

$$\begin{aligned} z_5 &= \frac{2+5i}{1-i} + \frac{2-5i}{1+i} = \frac{(2+5i)(1+i)+(2-5i)(1-i)}{(1+i)(1-i)} \\ &= \frac{2+2i+5i-5+2-2i-5i-5}{1^2-i^2} \\ &= -\frac{6}{2} = -3 \end{aligned}$$

Exercice 2.

1. $u = \frac{\sqrt{6}-i\sqrt{2}}{2}$ et $v = 1-i$

   on a

$$|u| = \frac{|\sqrt{6}-i\sqrt{2}|}{2} = \frac{\sqrt{6+2}}{2} = \frac{\sqrt{8}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

   et

$$u = \frac{\sqrt{6}-i\sqrt{2}}{2} = \sqrt{2}(\frac{\sqrt{3}}{2} - \frac{1}{2}i) = \sqrt{2}e^{\frac{-i\pi}{6}}$$

   donc

$$\begin{cases} |u| = \sqrt{2} \\ argu = \frac{-\pi}{6}. \end{cases}$$

2. On a
$$|v| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$
et
$$v = 1 - i = \sqrt{2}(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}) = \sqrt{2}e^{\frac{-i\pi}{4}}$$
donc
$$\begin{cases} |v| = \sqrt{2} \\ arg\ v = \frac{-\pi}{4}. \end{cases}$$

3. On en déduit que
$$\frac{u}{v} = \frac{\sqrt{2}e^{\frac{-i\pi}{6}}}{\sqrt{2}e^{\frac{-i\pi}{4}}} = e^{i(-\frac{i\pi}{6}+\frac{i\pi}{4})} = e^{\frac{i\pi}{12}}$$
par conséquent
$$\begin{cases} |\frac{u}{v}| = 1 \\ arg\ \frac{u}{v} = \frac{\pi}{12}. \end{cases}$$

Exercice 3. Notons $z_1 = 1 + i\sqrt{3}$. Alors le module de z_1 est $\sqrt{1^2 + \sqrt{3}^2} = \sqrt{1+3} = \sqrt{4} = 2$. Pour trouver un argument de z_1, cherchons θ tel que $\cos\theta = \frac{1}{2}$ et $\sin\theta = \frac{\sqrt{3}}{2}$. On trouve $\theta = \frac{\pi}{3}$ mod(2π). Ainsi
$$z_1 = 2e^{\frac{i\pi}{3}}.$$
De même, notons $z_2 = 1 - i$ Alors le module de z_2 est $\sqrt{1^2 + (-1)^2} = \sqrt{2}$. Pour trouver un argument de z_1, cherchons θ tel que $\cos\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ et $\sin\theta = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$. On trouve $\theta = -\frac{\pi}{4}$ mod(2π). Ainsi
$$z_2 = \sqrt{2}e^{-\frac{i\pi}{4}}.$$
Le quotient $\frac{1+i\sqrt{3}}{1-i}$ s'écrit donc $\frac{1+i\sqrt{3}}{1-i} = \frac{2e^{\frac{i\pi}{3}}}{\sqrt{2}e^{-\frac{i\pi}{4}}} = \sqrt{2}e^{i(\frac{\pi}{3}-(-\frac{\pi}{4}))} = \sqrt{2}e^{\frac{i7\pi}{12}}$.
Alors
$$z = (\frac{1+i\sqrt{3}}{1-i})^{20} = (\sqrt{2}e^{\frac{i7\pi}{12}})^{20} = (\sqrt{2})^{20}e^{\frac{i140\pi}{12}} = 2^{10}e^{\frac{i140\pi}{12}} = 1024e^{\frac{i140\pi}{12}}$$
Comme $140 = 6 \times 24 - 4$, on aura $\frac{140\pi}{12} = 6 \times 2\pi - \frac{\pi}{3} \equiv -\frac{\pi}{3} \mod(2\pi)$.
Ainsi
$$\frac{i140\pi}{12} = e^{-\frac{i\pi}{3}} = \cos(-\frac{i\pi}{3}) + i\sin(-\frac{i\pi}{3}) = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$
Enfin
$$z = 1024(\frac{1}{2} - i\frac{\sqrt{3}}{2}) = 512(1 - i\sqrt{3}).$$

Exercice 4.

a)

$$\begin{aligned} u^4 = -4 &\iff \begin{cases} |u^4| = |-4| \\ arg(u^4) = arg(-4) + 2k\pi, \quad k \in \mathbb{Z} \end{cases} \\ &\iff \begin{cases} |u|^4 = 4 \\ 4arg(u) = \pi + 2k\pi, \quad k \in \mathbb{Z} \end{cases} \\ &\iff \begin{cases} |u| = 4^{\frac{1}{4}} = (2^2)^{\frac{1}{4}} = \sqrt{2} \\ arg(u) = \frac{\pi}{4} + k\frac{\pi}{2}, \quad k \in \{0,1,2,3\} \end{cases} \end{aligned}$$

il y a quatre solutions:
$$u_0 = \sqrt{2}e^{\frac{i\pi}{4}} = \sqrt{2}(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) = 1 + i$$

$$u_1 = \sqrt{2}e^{\frac{3i\pi}{4}} = \sqrt{2}(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) = -1+i$$

$$u_2 = \sqrt{2}e^{\frac{5i\pi}{4}} = \sqrt{2}(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}) = -1-i = \overline{u_1}$$

$$u_3 = \sqrt{2}e^{\frac{7i\pi}{4}} = \sqrt{2}(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}) = 1-i = \overline{u_0}$$

b)

$$(z+1)^4 + 4(z-1)^4 = 0 \Longleftrightarrow (z+1)^4 = -4(z-1)^4 \Longleftrightarrow (\frac{z+1}{z-1})^4 = -4$$

On pose $u = \frac{z+1}{z-1}$, il y a 4 solutions que l'on trouve en exprimant z en fonction de u

$$\begin{aligned} u = \tfrac{z+1}{z-1} &\Longleftrightarrow u(z-1) = z+1 \\ &\Longleftrightarrow uz - u = z+1 \\ &\Longleftrightarrow z(u-1) = u+1 \\ &\Longleftrightarrow z = \tfrac{u+1}{u-1} \end{aligned}$$

Donc les solutions sont:

$$z_0 = \frac{u_0+1}{u_0-1} = \frac{1+i+1}{1+i-1} = \frac{2+i}{i} = 1-2i$$

$$z_1 = \frac{u_1+1}{u_1-1} = \frac{-1+i+1}{-1+i-1} = \frac{i}{-2+i} = \frac{i(-2-i)}{(-2)^2-i^2} = \frac{1}{5} - \frac{2}{5}i$$

$$z_2 = \frac{u_2+1}{u_2-1} = \frac{\overline{u_1}+1}{\overline{u_1}-1} = \overline{z_1} = \frac{1}{5} + \frac{2}{5}i$$

$$z_3 = \frac{u_3+1}{u_3-1} = \frac{\overline{u_0}+1}{\overline{u_0}-1} = \overline{z_0} = 1+2i$$

c) On a

$$(\sqrt{3}+i)^n = (2(\frac{\sqrt{3}}{2} + i\frac{1}{2}))^n = 2^n(e^{\frac{i\pi}{6}})^n = 2^n e^{\frac{ni\pi}{6}}$$

Donc

$$\begin{aligned} (\sqrt{3}+i)^n \in I\!R &\Longleftrightarrow (\sqrt{3}+i)^n - \overline{(\sqrt{3}+i)^n} = 0 \\ &\Longleftrightarrow e^{\frac{ni\pi}{6}} - e^{\frac{-ni\pi}{6}} = 0 \\ &\Longleftrightarrow sin(\frac{n\pi}{6}) = 0 \\ &\Longleftrightarrow \exists k \in Z\!\!Z, \quad \frac{n\pi}{6} = k\pi \\ &\Longleftrightarrow \exists k \in Z\!\!Z, \quad n = 6k \end{aligned}$$

et

$$\begin{aligned} (\sqrt{3}+i)^n \in iI\!R &\Longleftrightarrow (\sqrt{3}+i)^n + \overline{(\sqrt{3}+i)^n} = 0 \\ &\Longleftrightarrow e^{\frac{ni\pi}{6}} + e^{\frac{-ni\pi}{6}} = 0 \\ &\Longleftrightarrow cos(\frac{n\pi}{6}) = 0 \\ &\Longleftrightarrow \exists k \in Z\!\!Z, \quad \frac{n\pi}{6} = \frac{\pi}{2} + k\pi \\ &\Longleftrightarrow \exists k \in Z\!\!Z, \quad n = 3+6k \end{aligned}$$

Exercice 5.

a) On cherche les complexes tels que

1)

$$z^n = -i \iff \begin{cases} |z^n| = |-i| \\ arg(z^n) = arg(-i) + 2k\pi, \quad k \in \mathbb{Z} \end{cases}$$
$$\iff \begin{cases} |z|^n = 1 \\ n\ arg(z) = -\frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z} \end{cases}$$
$$\iff \begin{cases} |z| = 1 \\ arg(z) = -\frac{\pi}{2n} + \frac{2k\pi}{n}, \quad k \in \{0, 1, \ldots, n-1\} \end{cases}$$

Les solutions sont:

$$z_k = e^{i(-\frac{\pi}{2n} + \frac{2k\pi}{n})}, \quad k \in \{0, 1, \ldots, n-1\}$$

2)

$$z^n = 1 + i = \sqrt{2}(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) = \sqrt{2}e^{\frac{i\pi}{4}} \iff \begin{cases} |z^n| = \sqrt{2} \\ arg(z^n) = \frac{\pi}{4} + 2k\pi, \quad k \in \mathbb{Z} \end{cases}$$
$$\iff \begin{cases} |z|^n = \sqrt{2} = 2^{\frac{1}{2}} \\ n\ arg(z) = \frac{\pi}{4} + 2k\pi, \quad k \in \mathbb{Z} \end{cases}$$
$$\iff \begin{cases} |z| = 2^{\frac{1}{2n}} \\ arg(z) = \frac{\pi}{4n} + \frac{2k\pi}{n}, \quad k \in \{0, 1, \ldots, n-1\} \end{cases}$$

Les solutions sont:

$$z_k = 2^{\frac{1}{2n}} e^{i(\frac{\pi}{4n} + \frac{2k\pi}{n})}, \quad k \in \{0, 1, \ldots, n-1\}$$

b) $z^2 - z + 1 - i = 0$:

Le disciminant vaut $\Delta = (-1)^2 - 4(1 - i) = -3 + 4i = 1 + 4i - 4 = (1 + 2i)^2$

il y a deux solutions

$$z_1 = \frac{1 - (1 + 2i)}{2} = -i \quad et \quad z_2 = \frac{1 + (1 + 2i)}{2} = 1 + i$$

c) $z^{2n} - z^n + 1 - i = 0$:

On pose $Z = z^n$ alors

$$z^{2n} - z^n + 1 - i = 0 \iff Z^2 - Z + 1 - i = 0$$
$$\iff \begin{cases} Z = -i \\ Z = 1 + i \end{cases}$$
$$\iff \begin{cases} z^n = -i \\ z^n = 1 + i. \end{cases}$$

L'ensemble des solutions est d'après la qustion 1) est:

$$\{z_k = e^{i(-\frac{\pi}{2n} + \frac{2k\pi}{n})}, \quad k \in \{0, 1, \ldots, n-1\}; \ z_k = 2^{\frac{1}{2n}} e^{i(\frac{\pi}{4n} + \frac{2k\pi}{n})}, \quad k \in \{0, 1, \ldots, n-1\}\}$$

Exercice 6. Soit l'équation

$$z^3 - iz + 1 - i = 0 \quad (E)$$

a) Posons $z = a \in \mathbb{R}$,

$$(E) \iff a^3 + 1 - i(a + 1) = 0$$
$$\iff \begin{cases} a^3 + 1 = 0 \\ a + 1 = 0 \end{cases}$$
$$\iff a = -1$$

b) On peut diviser $z^3 - iz + 1 - i$ par $z + 1$

$$\begin{array}{llll|l} z^3 & & -iz & +1-i & z+1 \\ \hline z^3 & +z^2 & & & z^2 - z + 1 - i \\ \hline & -z^2 & -iz & +1-i & \\ & -z^2 & -z & & \\ \hline & & (1-i)z & +1-i & \\ & & (1-i)z & +1-i & \\ \hline & & & 0 & \end{array}$$

Donc $z^3 - iz + 1 - i = (z+1)(z^2 - z + 1 - i)$ et d'après l'exercice 5) $z^3 - iz + 1 - i = 0$ à pour solution $z_1 = -i$ et $z_2 = 1 + i$. donc les solutions de (E) sont:

$$\{-1, -i, 1+i\}$$

Exercice 7. Si on pose $Z = z^3$ donc

$$z^6 - iz^3 - 1 - i = 0 \Longleftrightarrow Z^2 - iZ - 1 - i = 0$$

Le disciminant vaut $\Delta = (-i)^2 - 4(-1-i) = 4 + 4i - 1 = (2+i)^2$
Les solutions de $Z^2 - iZ - 1 - i = 0$ sont

$$Z_1 = \frac{i + 2 + i}{2} = 1 + i \quad \text{et} \quad Z_2 = \frac{i - (2+i)}{2} = -1$$

Les solutions de $z^6 - iz^3 - 1 - i = 0$ vérifient

$$\begin{aligned} z^3 = 1 + i = \sqrt{2}e^{\frac{i\pi}{4}} \quad &\Longleftrightarrow \begin{cases} |z^3| = \sqrt{2} \\ argz^3 = \frac{\pi}{4} + 2k\pi, \quad k \in \mathbb{Z} \end{cases} \\ &\Longleftrightarrow \begin{cases} |z|^3 = 2^{\frac{1}{2}} \\ 3argz = \frac{\pi}{4} + 2k\pi, \quad k \in \mathbb{Z} \end{cases} \\ &\Longleftrightarrow \begin{cases} |z| = 2^{\frac{1}{6}} \\ argz = \frac{\pi}{12} + \frac{2k\pi}{3}, \quad k \in \{0,1,2\} \end{cases} \\ &\Longleftrightarrow \quad z \in \{2^{\frac{1}{6}}e^{\frac{i\pi}{12}},\ 2^{\frac{1}{6}}e^{\frac{3i\pi}{4}},\ 2^{\frac{1}{6}}e^{\frac{17i\pi}{12}}\} \end{aligned}$$

où

$$\begin{aligned} z^3 = -1 \quad &\Longleftrightarrow \begin{cases} |z^3| = 1 \\ argz^3 = \pi + 2k\pi, \quad k \in \mathbb{Z} \end{cases} \\ &\Longleftrightarrow \begin{cases} |z|^3 = 1 \\ 3argz = \pi + 2k\pi, \quad k \in \mathbb{Z} \end{cases} \\ &\Longleftrightarrow \begin{cases} |z| = 1 \\ argz = \frac{\pi}{3} + \frac{2k\pi}{3}, \quad k \in \{0,1,2\} \end{cases} \\ &\Longleftrightarrow \quad z \in \{e^{\frac{i\pi}{3}},\ -1,\ e^{\frac{-i\pi}{3}}\} \end{aligned}$$

Finalement il y a six solutions:

$$\{2^{\frac{1}{6}}e^{\frac{i\pi}{12}},\ 2^{\frac{1}{6}}e^{\frac{3i\pi}{4}},\ 2^{\frac{1}{6}}e^{\frac{17i\pi}{12}},\ e^{\frac{i\pi}{3}},\ -1,\ e^{\frac{-i\pi}{3}}\}$$

Exercice 8.

1. Linéariser $cos^5(x)$, $sin^5(x)$ et $cos^2(x)sin^3(x)$: Appliquant la formule d'Euler, on a

a)

$$\cos^5 x = (\frac{e^{ix}+e^{-ix}}{2})^5 = \frac{1}{32}(e^{ix}+e^{-ix})^5$$

d'après la formule de binôme on a

$$\begin{aligned}
\cos^5 x &= \frac{1}{32}\sum_{k=0}^{5} C_5^k (e^{ix})^k (e^{-ix})^{5-k} \\
&= \frac{1}{32}(e^{i5x}+5e^{i3x}+10e^{ix}+10e^{-ix}+5e^{-i3x}+e^{-i5x}) \\
&= \frac{1}{16}[\frac{e^{i5x}+e^{-i5x}}{2}+5\frac{e^{i3x}+e^{-i3x}}{2}+10\frac{e^{ix}+e^{-ix}}{2}] \\
&= \frac{1}{16}(\cos 5x+5\cos 3x+10\cos x)
\end{aligned}$$

b)

$$\sin^5 x = (\frac{e^{ix}-e^{-ix}}{2i})^5 = \frac{(e^{ix}+(-e^{-ix}))^5}{32i}$$

donc

$$\begin{aligned}
\sin^5 x &= \frac{1}{32i}\sum_{k=0}^{5} C_5^k (e^{ix})^k (-e^{-ix})^{5-k} \\
&= \frac{1}{32i}(e^{i5x}-5e^{i3x}+10e^{ix}-10e^{-ix}+5e^{-i3x}-e^{-i5x}) \\
&= \frac{1}{16}[\frac{e^{i5x}-e^{-i5x}}{2i}-5\frac{e^{i3x}-e^{-i3x}}{2i}+10\frac{e^{ix}-e^{-ix}}{2i}] \\
&= \frac{1}{16}(\sin 5x-5\sin 3x+10\sin x)
\end{aligned}$$

c)

$$\begin{aligned}
\cos^2 x\sin^3 x &= (\frac{e^{ix}+e^{-ix}}{2})^2(\frac{e^{ix}-e^{-ix}}{2i})^3 \\
&= \frac{e^{i2x}+2+e^{-i2x}}{4}\frac{e^{i3x}-3e^{ix}+3e^{-ix}-e^{-i3x}}{-8i} \\
&= \frac{e^{i5x}-3e^{i3x}-2e^{ix}+2e^{-ix}+e^{-i3x}-e^{-i5x}}{-32i} \\
&= -\frac{1}{16}[\frac{e^{i5x}-e^{-i5x}}{2i}-\frac{e^{i3x}-e^{-i3x}}{2i}-\frac{e^{ix}-e^{-ix}}{2i}] \\
&= -\frac{1}{16}\sin 5x+\frac{1}{16}\sin 3x+\frac{1}{8}\sin x
\end{aligned}$$

2. Soit $x \in \mathbb{R}$,

a) d'après la formule de Moivre et la formule du binôme:

$$\begin{aligned}
\cos(3x)+i\sin(3x) &= (\cos x+i\sin x)^3 \\
&= \cos^3 x+3i\sin x\cos^2 x-3\cos x\sin^2 x-i\sin^3 x
\end{aligned}$$

et il vient en identifiant les parties réel et imaginaires:

$$\cos 3x = \cos^3 x-3\cos x\sin^2 x$$

or $\sin^2 x = 1-\cos^2 x$, donc

$$\cos 3x = 4\cos^3 x-3\cos x$$

b) Il vient aussi

$$\sin 3x = 3\sin x\cos^2 x-\sin^3 x$$

comme $\cos^2 x = 1-\sin^2 x$, on obtient

$$\sin 3x = 3\sin x-4\sin^3 x$$

3. **a)** On utilise $\cos(2x) = 2\cos^2 x - 1$ donc

$$\begin{aligned}
\cos 2x + \cos x = -1 &\iff 2\cos^2 x - 1 + \cos x = -1 \\
&\iff \cos x(2\cos x + 1) = 0 \\
&\iff \cos x = 0 \text{ ou } \cos x = -\frac{1}{2} \\
&\iff x \equiv \frac{\pi}{2} \mod(\pi) \text{ ou} \\
&\quad\;\; x \equiv (\pi + \frac{\pi}{3}) \mod(2\pi) = \frac{4\pi}{3} \mod(2\pi) \text{ ou} \\
&\quad\;\; x \equiv -\frac{4\pi}{3} \mod(2\pi)
\end{aligned}$$

b) On utilise les calculs de la question précédente. On sait que $\cos(2x) = 2\cos^2 x - 1$ et $\cos 3x = 4\cos^3 x - 3\cos x$ donc

$$\begin{aligned}
\cos x + \cos 2x + \cos 3x = -1 &\iff 4\cos 3x + 2\cos^2 x + 4\cos x = 0 \\
&\iff 4\cos^3 x + 2\cos^2 x - 2\cos x = 0 \\
&\iff \cos x(2\cos^2 x + \cos x - 1) = 0
\end{aligned}$$

Afin de résoudre $2\cos^2 x + \cos x - 1 = 0$, on pose $X = \cos x$ et on cherche les racines de $2X^2 + X - 1 = 0$ qui sont $\frac{1}{2}$ et -1. Donc

$$2\cos^2 x + \cos x - 1 = 0 \iff \cos x = \frac{1}{2} \text{ ou } \cos x = -1$$

$$2\cos^2 x + \cos x - 1 = 0 \iff x \equiv \pm\frac{\pi}{3} \mod(2\pi) \text{ ou } x \equiv \pi \mod(2\pi)$$

Finalement les solutions de l'équation $\cos x + \cos 2x + \cos 3x = -1$ sont :

$$x \equiv \frac{\pi}{2} \mod(\pi),\; x \equiv \pm\frac{\pi}{3} \mod(2\pi) \text{ et } x \equiv \pi \mod(2\pi).$$